1 Overview

The Euler Characteristic (EC) densities for non-central T- and F-fields can be obtained by

$$\rho_i(u) = (-1)^{i-1} E \left[ \tilde{A}^{(i)} \det \left( \tilde{A}_{i-1} \right) | A = u, \tilde{A}_{i-1} = 0 \right] \theta_{i-1} \left( \tilde{A}_{i-1} = 0 \right)$$  \hspace{1cm} (1.1)

In order to use this formula, we first need to derive the first and second derivatives of these non-central fields. Derivation of the derivatives and the EC densities for central T- and F-random fields has been described in details by Worsley (1994). Since non-central random fields are derived from Gaussian random fields, we follow the derivation by Adler (1980) and Worsley (1994) to find the derivatives and EC densities for non-central T- and F-fields.

2 EC Densities of a Non-Central T-Field

2.1 Definition

Let $Z(t)$, $X_1(t)$, $X_2(t)$, $\cdots$, $X_m(t)$ be independent and identically distributed (iid) Gaussian random fields (Adler, 1980) at $t = (t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$, with zero mean and unit variance. For simplicity, the location index $t$ is dropped without loss of generality. The spatial correlation of these random fields can be expressed in terms of the variance of their first derivative $\Lambda = \text{var}(\tilde{Z}) = \text{var}(\tilde{X}_i)$ for $i = 1, 2, \cdots, m$. The squared-sum of $X_i$, $V = X_1^2 + X_2^2 + \cdots + X_m^2$, forms a $\chi^2$ random field with degrees of freedom $df = m$ (Adler, 1980; Worsley, 1994). Then

$$T = \frac{\sqrt{m}Z}{V^{\frac{1}{2}}}$$

is a T-field with $df = m$. Similarly, we define a non-central T-field as

$$S = \frac{\sqrt{m}(Z + \delta)}{V^{\frac{1}{2}}}$$

with $df = m$ and non-centrality parameter $nc = \delta$. In this work, $\delta$ is assumed to be a real-valued constant.
2.2 Derivatives

**Lemma 2.1.** The first and second derivatives of a non-central T-field \( S \) with \( df = m \) and \( nc = \delta \) can be expressed as

(a) \( \dot{S} = \sqrt{m} \left( 1 + \frac{s^2}{m} \right) W^{-\frac{1}{2}} R_1 \)

(b) \( \ddot{S} = \sqrt{m} \left( 1 + \frac{s^2}{m} \right) W^{-1} \left\{ W^{\frac{1}{2}} H - R_1 R_2^T - R_2 R_1^T + \frac{8}{\sqrt{m}} (2R_1 R_1^T - P) + \left( 1 + \frac{s^2}{m} \right)^{-\frac{1}{2}} W^{\frac{1}{2}} \delta \Lambda \right\} \)

where \( W \) is a non-central \( \chi^2 \) random variable with \( df = m + 1 \) and \( nc = \delta^2 \). \( R_1 \) and \( R_2 \) are iid \( \text{Norm} \{ 0, \Lambda \} \), \( H \sim \text{Norm}_{N \times N} \{ 0, M(\Lambda) \} \), and \( P \sim \text{Wishart}_N(\Lambda, m - 1) \). The elements of the matrix \( M(\Lambda) \) are given in Worsley (1994). In (a) and (b), equalities indicate equalities in law.

**Proof.** The first derivative of \( S \) can be written as

\[ \dot{S} = \frac{\sqrt{m} \dot{Z}}{V^{\frac{1}{2}}} - \frac{1}{2} \frac{\sqrt{m}(Z + \delta) \dot{V}}{V^{\frac{3}{2}}} . \]

Worsley (1994) showed that \( \dot{Z} \sim R_Z \) and \( \dot{V} \sim 2V^{\frac{1}{2}} R_V \) where \( R_Z, R_V \sim \text{iid Norm}_N(0, \Lambda) \). Thus we can write the above as

\[ \dot{S} = \frac{\sqrt{m}}{V} \left( V^{\frac{1}{2}} R_Z - (Z + \delta) R_V \right) . \]

Let \( W = V + (Z + \delta)^2 \), then \( W \sim \text{non-central \( \chi^2 \) with \( df = m + 1 \) and \( nc = \delta^2 \)}. Note that \( V^{-1} = \left( 1 + \frac{s^2}{m} \right) W^{-1} \), and \( R_1 = W^{\frac{1}{2}} (V^{\frac{1}{2}} R_Z - (Z + \delta) R_V) \sim \text{Norm}_N(0, \Lambda) \) since it is a linear combination of iid \( \text{Norm}_N(0, \Lambda) \). Using these facts, we can rewrite the above expression for \( \dot{S} \) to prove (a). For \( \ddot{S} \), we first take the derivative of \( S \)

\[ \ddot{S} = \frac{\sqrt{m} \ddot{Z}}{V^{\frac{1}{2}}} - \frac{1}{2} \frac{\sqrt{m} \dot{V} \dot{Z}^T}{V^{\frac{3}{2}}} - \frac{1}{2} \frac{\sqrt{m} \dot{Z} \dot{V}^T}{V^{\frac{3}{2}}} - \frac{1}{2} \frac{\sqrt{m}(Z + \delta) \ddot{V}}{V^{\frac{3}{2}}} + \frac{3}{4} \frac{\sqrt{m}(Z + \delta) \dot{V} \dot{V}^T}{V^{\frac{3}{2}}} . \]

Worsley (1994) showed that \( \dot{Z} | Z \sim (H_Z - Z \Lambda) \) and \( \dot{V} = 2 \left( P + R_V R_V^T - V \Lambda + V^{\frac{1}{2}} H_V \right) \) where \( H_Z, H_V \) are iid \( \text{Norm}_{N \times N}(0, M(\Lambda)) \). Thus, by rewriting the above in terms of \( R_Z, R_V, H_Z, \) and \( H_V \), we have

\[ \ddot{S} = \frac{\sqrt{m}}{V} \left( V^{\frac{1}{2}} H_Z - (Z + \delta) H_V \right) + \frac{\sqrt{m} \Lambda}{V^{\frac{3}{2}}} + \frac{\sqrt{m}}{V} \left( -R_V R_Z^T - R_Z R_V^T + 2 \frac{(Z + \delta)}{V^{\frac{3}{2}}} R_V R_V^T \right) - \frac{\sqrt{m}(Z + \delta) P}{V^{\frac{3}{2}}} . \]

Since \( (V^{\frac{1}{2}} H_Z - (Z + \delta) H_V) \) is a linear combination of normals, it is equivalent to \( W^{\frac{1}{2}} H \) where \( H \sim \text{Norm}_{N \times N}(0, M(\Lambda)) \). Next, let \( R_2 = W^{\frac{1}{2}} (V^{\frac{1}{2}} R_V - (Z + \delta) R_Z) \). Then \( R_1 \) and \( R_2 \) are iid \( \text{Norm}_N(0, \Lambda) \), and

\[ \frac{\sqrt{m}}{V} \left( -R_V R_Z^T - R_Z R_V^T + 2 \frac{(Z + \delta)}{V^{\frac{3}{2}}} R_V R_V^T \right) = \frac{\sqrt{m}}{V} \left( -R_1 R_1^T - R_2 R_2^T + 2 \frac{(Z + \delta)}{V^{\frac{3}{2}}} R_1 R_1^T \right) \]

(Worsley, 1994). Finally, since \( V^{-1} = \left( 1 + \frac{s^2}{m} \right) W^{-1} \), we can rewrite \( \ddot{S} \) above and prove (b). \( \square \)

The first derivative \( \dot{S} \) is almost identical to that of a T-field (Worsley, 1994), in which the non-central random variables \( S \) and \( W \) are replaced by the corresponding central random variables. The second derivative \( \ddot{S} \) is also similar to that of a T-field (Worsley, 1994), but has an extra term involving \( \delta \Lambda \).
2.3 EC Densities

**Theorem 2.1.** The $i$-dimensional EC density $\rho_i(u)$ of a non-central $T$-field $S$ with $df = m$ and $nc = \delta$ is given by

$$\rho_i(u) = (-1)^{i-1} \frac{(\det(\Lambda)^\frac{1}{2}}{(2\pi)^{\frac{i}{2}}} \sqrt{m} \left( 1 + \frac{u^2}{m} \right) Q_i,m(u),$$

where $Q_i,m(u)$ is a polynomial given by

$$Q_i,m(u) = \sum_{j=0}^{i \wedge} \frac{(-1)^{i+j+k-1}(i-1)!}{2^{j+k} j!k!} (\frac{u^2}{m})^{\frac{i-2j-k}{2}} \left( 1 + \frac{u^2}{m} \right)^{-\frac{k}{2}} \delta^k E \left[ W^{-\frac{i-2j-k}{2}} \right] f_S(u)$$

where $f_S(u)$ is the probability density function of $S$.

**Proof.** This is a result of (2.1). We derive this by conditioning (1.1) on $W$ and $S = u$, then by taking the expectation on $W$ (Worsley, 1994). In other words, we find

$$E \left[ \hat{S}^{(i)+} \det(\hat{S}_{i-1}) | S = u, \hat{S}_{i-1} = 0 \right] \theta_{i-1} \left( \hat{S}_{i-1} = 0 \right)$$

$$= EW \left[ E \left[ \hat{S}^{(i)+} \det(\hat{S}_{i-1}) | S = u, W, \hat{S}_{i-1} = 0 \right] \psi_{i-1} \left( \hat{S}_{i-1} = 0 | S = u, W \right) \right] f_S(S = u)$$

where $\psi_{i-1}$ is the conditional density function of $\hat{S}_{i-1}$ given $S$ and $W$. Since $\hat{S}$ is a multi-variate normal (Lemma 2.1),

$$\psi_{i-1} \left( \hat{S}_{i-1} = 0 | S, W \right) = \frac{1}{(2\pi)^{\frac{i-1}{2}} (\det(\Lambda_{i-1}))^{\frac{1}{2}} a^{i-1}}$$

where $a = \sqrt{m} \left( 1 + \frac{u^2}{m} \right) W^{-\frac{1}{2}}$. By conditioning on $S$, $W$, and $\hat{S}_{i-1} = 0$, we can express $\hat{S}_{i-1}$ as

$$\hat{S}_{i-1} = \frac{a}{b} (P_{i-1} + bH_{i-1} + d\Lambda_{i-1})$$

where $b = -\frac{\sqrt{m} W^{\frac{1}{2}}}{S}$, $d = b \left( 1 + \frac{u^2}{m} \right)^{-\frac{1}{2}} \delta$, $\Lambda_{i-1}$ is the first $i-1$ rows and columns of $\Lambda$, $P_{i-1} \sim Wishart_{i-1}(\Lambda_{i-1}, m-1)$, and $H_{i-1} \sim Norm_{i(i-1)}(0, M(\Lambda_{i-1}))$. This is independent of $\hat{S}$, thus we can split the expectation

$$E \left[ \hat{S}^{(i)+} \det(\hat{S}_{i-1}) | S, W, \hat{S}_{i-1} = 0 \right]$$

$$= E \left[ \hat{S}^{(i)+} | S, W, \hat{S}_{i-1} = 0 \right] E \left[ \det(\hat{S}_{i-1}) | S, W, \hat{S}_{i-1} = 0 \right]$$

The expectation of $\hat{S}^{(i)+}$ can be easily obtained as

$$E \left[ \hat{S}^{(i)+} | S, W, \hat{S}_{i-1} = 0 \right] = \frac{a}{(2\pi)^{\frac{i}{2}}}$$

where $\lambda_i = \operatorname{var}(R_{i,j}^{(i)}) | R_{i,1}^{(1)}, R_{i,2}^{(2)}, \ldots, R_{i,i}^{(i-1)}$, with $R_{i,j}^{(j)}$ indicating the $j$-th element of $R_{i}$ (Adler, 1980; Worsley, 1994). For the expectation of $\det(\hat{S}_{i-1})$, let $B$ be an orthogonal $(i-1) \times (i-1)$ matrix such that $B^T \Lambda_{i-1} B = I_{i-1}$, $B^T P_{i-1} B = P_{i-1}^* \sim Wishart_{i-1}(I_{i-1}, m-1)$, and $B^T H_{i-1} B = H_{i-1}^* \sim Norm_{(i-1)(i-1)}(0, M(I_{i-1}))$ (Worsley, 1994). Then

$$E \left[ \det(\hat{S}_{i-1}) | S, W, \hat{S}_{i-1} = 0 \right] = E \left[ \det \left( \frac{a}{b} (P_{i-1} + bH_{i-1} + d\Lambda_{i-1}) \right) \right]$$

$$= \det(\Lambda_{i-1}) \frac{a^{i-1}}{b^{i-1}} E \left[ \det \left( B^T (P_{i-1} + bH_{i-1} + d\Lambda_{i-1}) B \right) \right]$$

$$= \det(\Lambda_{i-1}) \frac{a^{i-1}}{b^{i-1}} E \left[ \det \left( P_{i-1}^* + bH_{i-1}^* + dI_{i-1} \right) \right].$$
Using the results from Lemma A.2 in Appendix A, we obtain

\[ E \left[ \det \left( \frac{\Lambda_{i-1}}{b} \right) \right] = \det(\Lambda_{i-1}) \frac{n-1}{\rho} \sum_{j=0}^{i-1} \frac{(-1)^j}{2^j j!} \sum_{k=0}^{i-2j-1} \frac{1}{k!} \frac{(m-1)(i-1)!}{i-2j-1} d^k. \]

Using these results, we can find inside the expectation \( E_W \cdot \hat{f} \).

\[
E \left[ \hat{S}^{(i)} + S, W, \hat{S}_{i-1} = 0 \right] \hat{E} \left[ \hat{S} \left| S, W, \hat{S}_{i-1} = 0 \right) \right] \psi_{i-1} \left( \hat{S}_{i-1} = 0 \mid S, W \right)
\]

\[
= \lambda_i (\det(\Lambda_{i-1}))^{\frac{j}{2}} \left( \frac{1}{(2\pi)^{\frac{j}{2}}} \sum_{j=0}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \frac{(-1)^j}{2^j j!} \sum_{k=0}^{i-2j-1} \frac{1}{k!} \frac{(m-1)(i-1)!}{i-2j-1} d^k \right).
\]

Note that \( \lambda_i \det(\Lambda_{i-1}) = \det(\Lambda) \). By substituting back \( a, b, \) and \( d, \) we have

\[
E \left[ \hat{S}^{(i)} + S, W, \hat{S}_{i-1} = 0 \right] \hat{E} \left[ \hat{S} \left| S, W, \hat{S}_{i-1} = 0 \right) \right] \psi_{i-1} \left( \hat{S}_{i-1} = 0 \mid S, W \right)
\]

\[
= \frac{(\det(\Lambda))^{\frac{j}{2}}}{(2\pi)^{\frac{j}{2}}} \sqrt{m} \left( 1 + \frac{S^2}{m} \right) \sum_{j=0}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \frac{(-1)^j}{2^j j!} \sum_{k=0}^{i-2j-1} \frac{1}{k!} \frac{(m-1)(i-1)!}{i-2j-1} \delta^{j}W_{-\frac{1}{2}-j-k}.
\]

Taking the expectation of the above at \( S = u \) over \( W, \) and multiplying by \((-1)^{i-1}\) and \( f_S(u), \) we prove the theorem.

From the theorem above, the EC densities \( \rho_i(u) \ (i = 0, 1, 2, 3) \) are given by

\[
\rho_0(u) = \int_{u}^{\infty} f_S(y)dy
\]

\[
\rho_1(u) = \frac{(\det(\Lambda))^{\frac{j}{2}}}{(2\pi)^{\frac{j}{2}}} \sqrt{m} \left( 1 + \frac{u^2}{m} \right) \left[ W^{-\frac{1}{2}} \right] f_S(u)
\]

\[
\rho_2(u) = \frac{(\det(\Lambda))^{\frac{j}{2}}}{2\pi} \sqrt{m} \left( 1 + \frac{u^2}{m} \right) \left\{ (m-1) \left( \frac{u^2}{m} \right) ^\frac{j}{2} E \left[ W^{-\frac{1}{2}} \right] - \left( 1 + \frac{u^2}{m} \right) ^\frac{j}{2} \left[ W^{-\frac{1}{2}} \right] \delta \right\} f_S(u)
\]

\[
\rho_3(u) = \frac{(\det(\Lambda))^{\frac{j}{2}}}{(2\pi)^{\frac{j}{2}}} \sqrt{m} \left( 1 + \frac{u^2}{m} \right) \left\{ (m-1)(m-2) \left( \frac{u^2}{m} \right) E \left[ W^{-\frac{3}{2}} \right] - 2(m-1) \left( \frac{u^2}{m} \right) ^\frac{j}{2} \left( 1 + \frac{u^2}{m} \right) ^\frac{j}{2} E \left[ W^{-\frac{1}{2}} \right] \delta \right.
\]

\[
\left. + \left( 1 + \frac{u^2}{m} \right) ^{-1} \left[ W^{-\frac{1}{2}} \right] \delta^2 - E \left[ W^{-\frac{1}{2}} \right] \delta \right\} f_S(u)
\]

where \( f_S(\cdot) \) is the probability density function of a non-central T-random variable with \( df = m \) and \( nc = \delta \) (Lehmann, 1986) given by

\[
f_S(s) = \frac{1}{2^{\frac{m+2}{2}} \Gamma \left( \frac{m+1}{2} \right) } \sqrt{m\pi} \int_{0}^{\infty} y^{\frac{m+1}{2}} \exp \left( -\frac{y}{2} \right) \exp \left( -\frac{1}{2} \left( \frac{s}{\sqrt{\frac{y}{m}} - \delta} \right) ^2 \right) dy.
\]
To find the expectation of a non-central $\chi^2$ random variable $W$, we can use the results from Johnson and Kotz (1970). For a non-central $\chi^2$ random variable $W$ with $df = \nu$ and $nc = \eta$, the expectation of $W^d$ with real-valued $d$ is given by

$$E[W^d] = 2^d \Gamma\left(d + \frac{\nu}{2}\right) \sum_{j=0}^{\infty} \frac{\Gamma(d + 1)}{\Gamma(j + 1)\Gamma(d - j + 1)} \left(\frac{\eta}{2}\right)^j \frac{1}{\Gamma\left(j + \frac{d}{2}\right)}.$$  \hspace{1cm} (2.1)

Note that, for $\delta = 0$, the EC densities above are equal to that of a T-field with the same $df$.

3 EC Densities of a Non-Central F-Field

3.1 Definition

Let $Z_1, Z_2, \ldots, Z_m$, and $X_1, X_2, \ldots, X_n$ be iid Gaussian random fields at $t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N$, with zero mean and unit variance, and $\Lambda = \text{var}(\dot{Z}_j) = \text{var}(X_k)$. Let

$$U = \sum_{j=1}^{m} (Z_j + \eta_j)^2$$

with real valued scalars $\eta_j$, and let

$$V = X_1^2 + X_2^2 + \cdots + X_n^2.$$  \hspace{1cm} (2.1)

Then we define $U$ as a non-central $\chi^2$ random field with $df = m$ and $nc = \eta = \sum_{j=1}^{m} \eta_j^2$, and $V$ is a $\chi^2$ random field with $df = n$. Furthermore, we define $F = \frac{U}{V}$ as a non-central $F$-random field with $df = (m, n)$ and $nc = \eta$.

Before we derive derivatives and EC densities of the non-central field $F$, we first find derivatives of a non-central $\chi^2$ field. Then we find the derivatives and EC densities for $G = \frac{U}{V} = \frac{m}{n} F$ for simplicity, and transform the results back to $F$ (Worsley, 1994).

3.2 Derivatives

Lemma 3.1. The first and second derivatives of a non-central $\chi^2$ field $U$ with $df = m$ and $nc = \eta$ can be expressed as

(a) $\dot{U} = 2U^\frac{1}{2} R_U$

(b) $\ddot{U} = 2 \left( P + R_U R_U^T - U\Lambda + D\Lambda + U^\frac{1}{2} H \right)$

where $R_U \sim \text{Norm}_N(0, \Lambda)$, $P \sim \text{Wishart}_N(\Lambda, m-1)$, $D \sim \text{Norm}(\eta, \eta)$, and $H \sim \text{Norm}_{N \times N}(0, M(\Lambda))$. The equalities above are equalities in law.

Proof. The first derivative of $U$ can be written as $\dot{U} = 2 \sum_{i=1}^{m} (Z_i + \eta_i) \dot{Z}_i$. From Worsley (1994), $\dot{Z}_i \sim \text{Norm}_N(0, \Lambda)$. Thus $\dot{U}$ is a sum of iid normals with mean 0 and variance $4U$, and can be written as $\dot{U} = 2U^\frac{1}{2} R_U$, with $R_U \sim \text{Norm}_N(0, \Lambda)$. This proves (a). The second derivative can be written as

$$\ddot{U} = \left( \sum_{i=1}^{m} (Z_i + \eta_i) \right)' \dot{Z}_i = \sum_{i=1}^{m} \dot{Z}_i \dot{Z}_i^T + \sum_{i=1}^{m} (Z_i + \eta_i) \ddot{Z}_i.$$  \hspace{1cm} (2.1)

Since $\ddot{Z}_i \sim \text{Norm}_{N \times N}(-Z_i, \Lambda, M(\Lambda))$ (Worsley, 1994), the second sum above, $2 \sum_{i=1}^{m} (Z_i + \eta_i) \ddot{Z}_i$ is normally distributed with mean $-2U\Lambda + 2D\Lambda$ and variance $4U M(\Lambda)$, where $D = \sum_{i=1}^{m} (Z_i + \eta_i)$. It is easy to see that $D \sim \text{Norm}(\eta, \eta)$, and we can write

$$\ddot{U} = 2 \left( \sum_{i=1}^{m} \dot{Z}_i \dot{Z}_i^T - U\Lambda + D\Lambda + U^\frac{1}{2} H \right)$$

where $H \sim \text{Norm}_{N \times N}(0, M(\Lambda))$. The sum $\sum_{i=1}^{m} \dot{Z}_i \dot{Z}_i^T$ can be written as $P + R_U R_U^T$ where $P \sim \text{Wishart}_N(\Lambda, m-1)$ (Worsley, 1994), and this proves (b). \hfill \Box
The first derivative $\dot{U}$ is almost identical to that of a $\chi^2$ field (Worsley, 1994). The second derivative is also similar to that of a $\chi^2$ field, with an extra term $2DA$.

**Lemma 3.2.** Let $F$ be a non-central $F$-random field with $df = (m, n)$ and $nc = \eta$, and let $G = \frac{m}{n} F$. Then the first and second derivatives of $G$ are

(a) $\dot{G} = 2G\dot{\frac{1}{2}}(1 + G)W^{\frac{1}{2}} R_1$

(b) $\ddot{G} = 2(1 + G)$

\[
\left\{ W^{-1} \left( P - GQ + (1 + 3G)R_1 R_1^T - G\dot{\frac{1}{2}}(R_1 R_1^T + R_2 R_2^T) + DA \right) + G\dot{\frac{1}{2}}W^{-\frac{1}{2}} H \right\}
\]

where $R_1, R_2 \sim \text{ iid } \text{Norm}_N(\mathbf{0}, \Lambda)$, $P \sim \text{ Wishart}_N(\Lambda, m - 1)$, $Q \sim \text{ Wishart}_N(\Lambda, n - 1)$, $D \sim \text{ Norm}(\eta, \eta)$, and $W = U + V$ is a non-central $\chi^2$ field with $df = m + n$ and $nc = \eta$. The equalities above are equalities in law.

**Proof.** The proof for (a) is identical to that of the central $G$ (Worsley, 1994), which follows from the first derivative of $U$ in Lemma 3.1.

The proof for (b) is similar to that of a central $F$-field (1994). The only difference is that $\dot{U}$ has an extra term $2DA$. Thus the term $2(1 + G)W^{-1} DA$ is added to the second derivative of the central $G$ (1994), resulting in (b) above. \(\square\)

### 3.3 EC Densities

**Theorem 3.1.** The $i$-dimensional EC density $\rho_i(u)$ of a non-central $F$-field $F$ is given by

\[
\rho_i(u) = (-1)^{i-1} \left( \frac{(\det(\Lambda))^{\frac{1}{2}}}{(2\pi)^\frac{i}{2}} \right) 2 \left( 1 + \frac{m}{n} u \right) \left( \frac{m}{n} u \right)^{-\frac{i}{2} + 1} K_{i,m,n}(u)
\]

where $K_{i,m,n}(u)$ is a polynomial given by

\[
K_{i,m,n}(u) = \sum_{j=0}^{i-1} \sum_{k=0}^{i-2j-1} \sum_{\ell=0}^{i-1-j-2j-1} \left( \begin{array}{c} m - 1 \\ i - 2j - k - 1 \\ k - \ell \end{array} \right) \frac{(-1)^{i+j-k}(N-1)!}{2j!}
\]

\[
\left( \frac{m}{n} u \right)^{j+k-\ell} E[W^{-\frac{1}{2}}] E[D^{\ell}] \left( \frac{n}{m} \right) f_F(u)
\]

where $f_F(u)$ is the probability density function of $F$.

**Proof.** For simplicity, we find the expectation (1.1) for $G = \frac{m}{n} F = \frac{U}{V}$, then we transform it back to $F$. The proof is similar to that of Theorem 2.1. First, we write (1.1) as

\[
E \left[ \dot{G}^{(i)} \det \left( \dot{G} \right) \left| G = g, \dot{G} = \mathbf{0} \right\} \right] \theta_{i-1} \left( \dot{G} = \mathbf{0} \right)
\]

\[
= E_{\dot{W}} \left[ E \left[ \dot{G}^{(i)} \det \left( \dot{G} \right) \left| G = g, W, \dot{G} = \mathbf{0} \right\} \psi_{i-1} \left( \dot{G} = \mathbf{0} \right| G = g, W \right] \right] f_{\dot{G}}(G = g)
\]

where $\theta_{i-1}$ is the probability density function of $\dot{G}_{i-1}$, $\psi_{i-1}$ is the conditional density function of $\dot{G}_{i-1}$ conditioned upon $G$ and $W$, and $f_{\dot{G}}$ is the probability density function of $G$. Since $\dot{G}$ is a multi-variate normal,

\[
\psi_{i-1} \left( \dot{G} = \mathbf{0} \right| G, W \right) = \frac{1}{(2\pi)^{\frac{i}{2}} (bc)^{-1}(\det(\Lambda_{i-1}))^{\frac{1}{2}}}
\]

where $b = G\dot{\frac{1}{2}}W^{\frac{1}{2}}$ and $c = 2(1 + G)W^{-1}$. Conditioning on $G$, $W$, and $\dot{G}_{i-1} = \mathbf{0}$, we can write $\dot{G}_{i-1}$ as

\[
\dot{G}_{i-1} = c(P_{i-1} + aQ_{i-1} + bH_{i-1} + d\Lambda_{i-1})
\]
where \( a = -G = (-1)G \) and \( d = D \). Similar to the proof of Theorem 2.1, we can split the expectation above

\[
E \left[ \dot{G}^{(i)} \det \left( \tilde{G}_{i-1} \right) \right] G, W, \dot{G}_{i-1} = 0 = E \left[ \dot{G}^{(i)} \right] G, W, \dot{G}_{i-1} = 0 \right] E \left[ \det \left( \tilde{G}_{i-1} \right) \right] G, W, \dot{G}_{i-1} = 0 \right] .
\]

The expectation of \( \dot{G}^{(i)} \) is \( \frac{\lambda^2}{(2\pi)^{\frac{1}{2}}} \). For the expectation of \( \det(\dot{G}_{i-1}) \), we use the orthogonalization technique used in the proof of Theorem 2.1. Let \( B \) be an \((i-1) \times (i-1)\) orthogonal matrix described in the proof of Theorem 2.1. Then

\[
E \left[ \det(\dot{G}_{i-1}) \right] G, W, \dot{G}_{i-1} = 0 = \det(\Lambda_{i-1}) c^{i-1} E \left[ \det \left( P^*_{i-1} + aQ^*_{i-1} + bH^*_{i-1} + dI_{i-1} \right) \right]
\]

where \( P^* \) and \( H^* \) are as described in the proof of Theorem 2.1 and \( B^TQB = Q^* \sim \text{Wishart}_{i-1}(I_{i-1}, n-1) \). From Lemma A.4 of Appendix A, we can write the above as

\[
E \left[ \dot{G}_{i-1} \right] G, W, \dot{G}_{i-1} = 0 = \det(\Lambda_{i-1}) c^{i-1} \sum_{j=0}^{\frac{i-2}{2}} (-1)^j (i-1)! \frac{1}{2^j} \sum_{k=0}^{i-2j-1} \left( \begin{array}{c} m-1 \\ i-2j-k-1 \end{array} \right) \sum_{\ell=0}^{k} \frac{a^{k-\ell}E[d^\ell]}{\ell!} \left( \begin{array}{c} n-1 \\ k-\ell \end{array} \right).
\]

Using these results, we can find inside the expectation \( E_W[\cdot] \),

\[
E \left[ \dot{G}^{(i)} \right] G, W, \dot{G}_{i-1} = 0 = E \left[ \dot{G}_{i-1} \right] G, W, \dot{G}_{i-1} = 0 \psi_{i-1} \left( \dot{G}_{i-1} = 0 \right) G, W = \frac{(\det(\Lambda))^\frac{i}{2}}{(2\pi)^\frac{i}{2}} \frac{c}{b^{i-2}} \sum_{j=0}^{\frac{i-2}{2}} (-1)^j (i-1)! \frac{1}{2^j} \sum_{k=0}^{i-2j-1} \left( \begin{array}{c} m-1 \\ i-2j-k-1 \end{array} \right) \sum_{\ell=0}^{k} \frac{a^{k-\ell}E[d^\ell]}{\ell!} \left( \begin{array}{c} n-1 \\ k-\ell \end{array} \right).
\]

Substituting back \( a, b, c, \) and \( d \), we have

\[
E \left[ \dot{G}^{(i)} \right] G, W, \dot{G}_{i-1} = 0 = \phi_{i-1} \left( \dot{G}_{i-1} = 0 \right) G, W = \frac{(\det(\Lambda))^\frac{i}{2}}{(2\pi)^\frac{i}{2}} 2(1 + G)^{\frac{i}{2} + 1} \sum_{j=0}^{\frac{i-2}{2}} \sum_{k=0}^j \sum_{\ell=0}^{k} \left( \begin{array}{c} m-1 \\ i-2j-k-1 \end{array} \right) \left( \begin{array}{c} n-1 \\ k-\ell \end{array} \right) (-1)^{j+k-\ell} (i-1)! \frac{1}{2^j \ell!} G^{j+k-\ell} W^{\frac{i-2j}{2}} E[D^\ell].
\]

Taking the expectation \( E_W[\cdot] \) at \( G = g \), multiplying by \((\lambda)^{i-1}\) and \( f_G(g) \), and transforming \( G \) to \( F = \frac{m}{m-1} G \), we can obtain the EC density for \( F \).

\[\square\]

Note that, since \( D \sim \text{Norm}(\eta, \eta) \), \( E[D^0] = 1 \), \( E[D] = \eta \), and \( E[D^2] = \eta + \eta^2 = \eta(1 + \eta) \). The EC
det(Λ) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f_F(y) dy

\rho(c) = \frac{(\text{det}(\Lambda))^\frac{c}{2}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f_F(y) dy

\rho_1(u) = \frac{(\text{det}(\Lambda))^\frac{1}{2}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f_F(y) dy

\rho_2(u) = \frac{(\text{det}(\Lambda))^\frac{1}{2}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f_F(y) dy

\rho_3(u) = \frac{(\text{det}(\Lambda))^\frac{3}{2}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f_F(y) dy

where the probability density function of F, \( f_F(u) \) (Johnson and Kotz, 1970), is

\[ f_F(u) = \frac{1}{\beta^\frac{n}{2} \Gamma(n/2)} \frac{u^{n/2 - 1}}{(n + nu)^{n/2 + m/2}} \sum_{j=0}^{\infty} \left( \frac{\eta u}{n + nu} \right)^j \frac{\Gamma(n/2 + j)}{j! \Gamma((n + m)/2)} \frac{\Gamma((n + m)/2)}{\Gamma((n + m)/2) + j} \]

and \( E[W]\) is given by (2.1).

Appendix

A Matrix Algebra

We prove some results used in the earlier proofs. Let \( A \) be an \( N \times N \) matrix, then we denote \( \text{det} r_j(A) \) to be the sum of the determinant of all the \( j \times j \) principal minors of \( A \). Note that, for an \( N \)-dimensional identity matrix \( I_N \), \( \text{det} r_k(I_N) = \binom{N}{N-k} \). This is a result from the expansion of the characteristic polynomial \( \text{det}(I_N + xI_N) = (1 - x)^N \).

Lemma A.1. Let \( P \sim \text{Wishart}_N(I_N, m) \) and \( a \) be a real number. Then

\[ E[\text{det} r_j(P + aI_N)] = \sum_{k=0}^{j} \binom{N}{j-k} \frac{m!}{(m-j+k)!} a^k. \]

Proof. The proof is similar to that of Lemma A.4, Worsley (1994).

\[ E[\text{det} r_j(P + aI_N)] = \sum_{k=0}^{j} \binom{N-k}{j-k} \frac{m!}{(m-j+k)!} \text{det} r_k(I_N) a^k \]

\[ = \sum_{k=0}^{j} \binom{N-k}{j-k} \frac{m!}{(m-j+k)!} \binom{N}{N-k} a^k \]

\[ = \sum_{k=0}^{j} \frac{(N-k)!m!N!}{(j-k)!(N-j)!(m-j+k)!(N-k)!} a^k \]

\[ = \sum_{k=0}^{j} \frac{m!}{j-k} \frac{N!}{(N-j)!} a^k. \]
Lemma A.2. Let $H \sim \text{Norm}_{N \times N}(\mathbf{0}, M(I_N))$. Then for real numbers $a$ and $b$,

$$E[\det(P + aI_N + bH)] = \sum_{j=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{(-1)^{j} j!}{2^{j} j!} \sum_{k=0}^{N-2j} \left( \binom{m}{N-2j-k} \frac{k!}{k!} a^k. \right)$$

Proof. By fixing $P + aI$, we can use Lemma A.2, Worsley (1994).

$$E[\det(P + aI_N + bH)] = \sum_{j=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{(-1)^{j} (2j)!}{2^{j} j!} b^{2j} E[\det r_{N-2j}(P + aI_N)]$$

and the rest follows from Lemma A.1.

Lemma A.3. Let $Q \sim \text{Wishart}_{N}(I_N, n)$, and $c$ be a real number. Then

$$E[\det r_j(P + aQ + cI_N)] = \sum_{k=0}^{j} \left( \frac{m}{j-k} \right) \frac{1}{(N-j)!} \sum_{\ell=0}^{k} a^{k-\ell} c^{\ell} \left( \frac{n}{k-\ell} \right) \frac{N!}{\ell!}.$$

Proof. Holding $aQ + cI_N$ fixed, then using Lemma A.1, we can show

$$E[\det r_j(P + aQ + cI_N)]$$

$$= \sum_{k=0}^{j} \left( \frac{m}{j-k} \right) \frac{1}{(N-j)!} \sum_{\ell=0}^{k} a^{k-\ell} c^{\ell} \left( \frac{n}{k-\ell} \right) \frac{N!}{\ell!}$$

Lemma A.4. Let $P$, $Q$, $H$, and $I_N$ be as defined above, and let $a$, $b$, and $c$ be real numbers. Then

$$E[\det(P + aQ + bH + cI_N)]$$

$$= \sum_{j=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{(-1)^{j} j!}{2^{j} j!} b^{2j} E[\det r_{N-2j}(P + aQ + cI_N)]$$

Proof. Holding $P + aQ + cI_N$ as a constant, then using Lemma A.3, we can show

$$E[\det(P + aQ + bH + cI_N)]$$

$$= \sum_{j=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \frac{(-1)^{j} (2j)!}{2^{j} j!} b^{2j} \sum_{k=0}^{N-2j} \left( \frac{m}{N-2j-k} \right) \frac{1}{(2j)!} \sum_{\ell=0}^{k} a^{k-\ell} c^{\ell} \left( \frac{n}{k-\ell} \right) \frac{N!}{\ell!}$$

9
References


